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Weyl Equation in Some Anisotropic Stiff Fluid Universes

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The Weyl equation (massless Dirac equation) is studied in a family of exact solutions of the Einstein equations whose material content is a perfect fluid with stiff equation of state ($p = \varepsilon$) and which are of Bianchi type I. The field equation is solved exactly for some members of the family.

1. INTRODUCTION

Recently there has been an increasing interest in quantum mechanics in external gravitational fields. Several authors have presented the analysis of the spectrum of the hydrogen atom in some particular metrics (Audretsch and Schäfer, 1978*a,b*) as well as in general gravitational fields (Parker, 1980*a,b*; Parker and Pimentel, 1982) and also as possible detectors of gravitational waves (Leen *et al.*, 1983). The detailed study of the exact solutions of the relativistic equations in curved spacetimes is a prerequisite to the construction of a quantum field theory in those curved backgrounds. The reported exact solutions to the Dirac and Weyl equations have been obtained by the method of separation of variables [for recent exact solutions see Cimento and Mollerach (1986, 1987), Barut and Duru (1987), Krori *et al.* (1988), Percoco and Villalba (1991), Villalba and Percoco (1990), Villalba (1990), and Castagnino *et al.* (1988)]. The question of the separability of the Dirac equation in curved spacetimes has been considered by several authors (Iyer and Kamaran, 1991; Iyer and Vishveshwara, 1987; Shishkin and Villalba, 1989). Here we consider the massless spinor field equation in a family of stiff fluid solutions. The metric was obtained by Jacobs (1968) and is also given by Vajk and Eltgroth (1970) and it is a particular case of

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the metrics studied by Thorne (1967); more recently it was rediscovered by Iyer and Vishveshwara (1987) while looking for exact solutions of Einstein's equations in which the Dirac equation separates. We write the metric in the "Kasner-like" form

$$ds^2 = dt^2 - a_1^2(t)\, dx^2 - a_2^2(t)\, dy^2 - a_3^2(t)\, dz^2 \tag{1}$$

where

$$a_1(t) = a_2(t) = t^q, \qquad a_3 = t^{1-2q} \tag{2}$$

This metric is a one-parameter family of solutions to Einstein's equations with a perfect stiff fluid. The parameter q is related to the energy density by the relation

$$\varepsilon = p = \frac{q(2-3q)}{t^2} \tag{3}$$

The qualitative features of the expansion depend on q in the following way: for $\frac{1}{2} < q$, the universe expands from a "cigar" singularity; for $q = \frac{1}{2}$, the universe expands purely transversely from an initial "barrel" singularity; for $0 < q < \frac{1}{2}$, the initial singularity is "point-like" if $q \leq 0$, we have a "pancake" singularity. The case $q = \frac{1}{3}$ is the isotropic universe with a stiff fluid; the case $q = 0$ is a region of the Minkowski spacetime in non-Cartesian coordinates. This family of metrics is "Kasner-like" in the sense that the sum of the exponents is equal to one, but the sum of the squares is not equal to one except in the two cases when $q = 0$ and $q = \frac{2}{3}$, when we have the vacuum. The symmetries of these spacetimes can be described by four spacelike Killing vector fields,

$$\xi_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad \xi_2 = \frac{\partial}{\partial x}, \qquad \xi_3 = \frac{\partial}{\partial y}, \qquad \xi_4 = \frac{\partial}{\partial z} \tag{4}$$

The first vector corresponds to the rotational symmetry in the plane xy and the other three to the translational symmetries along the x, y, and z axes. The nonvanishing commutators are

$$[\xi_1, \xi_2] = \xi_3, \qquad [\xi_3, \xi_1] = -\xi_2 \tag{5}$$

It is the high symmetry of this spacetime that makes it possible to separate the variables, and in some cases to solve exactly the field equations.

2. FIELD EQUATION

In order to write the field equation for a massless spin-$\frac{1}{2}$ field we introduce a tetrad $e^\alpha_\mu(x)$ that satisfies the relation

$$g_{\mu\nu}(x) = e^\alpha{}_\mu(x)\, e^\beta{}_\nu(x)\, \eta_{\alpha\beta} \tag{6}$$

For the present case we can choose

$$e^0{}_0(x) = 1, \qquad e^i{}_i(x) = a_i(t), \qquad e^0{}_i(x) = 0 \tag{7}$$

where the a_i are given by equation (2). A set of curved spacetime Dirac matrices that satisfy

$$\gamma_\mu(x)\gamma_\nu(x) + \gamma_\nu(x)\gamma_\mu(x) = -2g_{\mu\nu}(x) \tag{8}$$

is easily calculated (Chimento and Mallerach, 1986; Castagnino *et al.*, 1988)

$$\gamma^0 = \tilde{\gamma}^0, \qquad \gamma^i = a_i^{-1}(t)\tilde{\gamma}^i, \qquad \gamma_0\tilde{\gamma}_0, \qquad \gamma_i = a_i(t)\tilde{\gamma}_i \tag{9}$$

where the $\tilde{\gamma}^\alpha$ are Dirac matrices in flat Minkowski spacetime.

The Weyl equation in curved spacetime is

$$\gamma^\mu \nabla_\mu \psi(x) = 0 \tag{10a}$$

$$(1 + \gamma^5)\psi = 0 \tag{10b}$$

with

$$\nabla_\mu = \partial_\mu + \sigma_\mu, \qquad \sigma_\mu = \tfrac{1}{8}[\tilde{\gamma}^\alpha \tilde{\gamma}^\beta] e^\nu_\alpha e_{\beta\nu;\mu} \tag{11a}$$

and

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 \tag{11b}$$

In the case of the metric (1) we can separate variables according to

$$\psi_{\mathbf{k}} = \frac{1}{(2\pi)^{3/2}(a_1a_2a_3)^{1/2}} \begin{matrix} f_{\mathbf{k}1}(t) \\ f_{\mathbf{k}2}(t) \\ f_{\mathbf{k}3}(t) \\ f_{\mathbf{k}4}(t) \end{matrix} \; e^{i\mathbf{x}\cdot x} \tag{12}$$

Substituting (12) into equation (10a), we have

$$\frac{\partial f_{\mathbf{k}}(t)}{\partial t} := \dot{f}_{\mathbf{k}} = -iA_{\mathbf{k}}(t) f_{\mathbf{k}}(t) \tag{13}$$

where the matrix $A_{\mathbf{k}}$ is

$$A_{\mathbf{k}} = \begin{matrix} 0 & 0 & -K_3 & -K^- \\ 0 & 0 & -K^+ & K_3 \\ -K_3 & -K^- & 0 & 0 \\ -K^+ & K_3 & 0 & 0 \end{matrix}$$

and

$$K_i := k_i/a_i(t), \qquad K^\pm = K_1 \pm iK_2 \tag{14}$$

Equation (10b) gives the conditions $f_{k1} = f_{k3} := F(t)$, $f_{k2} = f_{k4} := G(t)$. Taking into account this condition in equation (13) gives

$$\dot{F} = i(K^- G + K_3 F) \tag{15a}$$

$$\dot{G} = i(K^+ F - K_3 G) \tag{15b}$$

After elimination of $G(t)$ or $F(t)$ we obtain second-order differential equations for $F(t)$ and $G(t)$. The differential equations are, after using the explicit form of the $a_i(t)$,

$$t^q \ddot{F} + q t^{q-1} \dot{F} + [ik_3(1-3q)t^{3q-2} + k_\perp^2 t^{-q} + k_3^2 t^{5q-2}]F = 0 \tag{16a}$$

$$t^q \ddot{G} + q t^{q-1} \dot{G} + [ik_3(1-3q)t^{3q-2} + k_\perp^2 t^{-q} + k_3^2 t^{5q-2}]G = 0 \tag{16b}$$

where $k_\perp^2 = k_1^2 + k_2^2$. It is clear that we need to solve only one of these equations, say (16a), and then the functional form of $G(t; k_3)$ will be that of $F(t; -k_3)$. There are several values of q for which equation (16) can be solved; they are considered in the following section.

3. EXACT SOLUTIONS

Here we consider those values of q for which it is possible to have exact solutions to equations (16).

$q = 0$

As mentioned above, this is a region of flat spacetime in non-Cartesian coordinates, as can be seen from the coordinate transformation $t' = t \sinh z$, $x' = x$, $y' = y$, $z' = t \cosh z$. The field equation is

$$\ddot{F} + \left(\frac{k_3^2 + ik_3}{t^2} + k_\perp^2 \right) F = 0 \tag{17}$$

the solution is

$$F = \sqrt{t}\, [A_k J_\nu(k_\perp t) + B_k J_{-\nu}(k_\perp t)] \tag{18}$$

where

$$\nu = \frac{[1 - 4(k_3^2 + ik_3)]^{1/2}}{2}$$

and J_ν is the Bessel function of order ν. The Dirac equation in this flat Kasner spacetime was considered by Shishkin and Andrushkevich (1985).

$q = \frac{1}{3}$

This case is a flat Robertson-Walker universe with $t^{1/3}$ expansion law, and now the field equation is

$$t^2\ddot{F} + \frac{t}{3}\dot{F} + k_3^2 t^{4/3} F = 0, \qquad k^2 = k_\perp^2 + k_3^2 \tag{19}$$

The solution is

$$F = t^{1/3}\left[A_k J_{1/2}\left(\frac{3k}{9}t^{2/3}\right) + B_k J_{-1/2}\left(\frac{3k}{9}t^{2/3}\right)\right] \tag{20}$$

where J_ν is the Bessel function of order ν. The Weyl equation in Robertson-Walker metrics with arbitrary expansion law was considered by Villalba and Percoco (1990).

$q = \frac{1}{2}$

The field equation takes the form

$$t^2\ddot{F} + \frac{t}{2}\dot{F} + [(k_\perp^2 + ik_3)t + k_3^2 t^2]F = 0 \tag{21}$$

with the solutions

$$F = t^{-1/4}[A_k W_{\kappa,1/4}(2ik_3 t) + B_k W_{-\kappa,1/4}(-2ik_3 t)] \tag{22}$$

where $W_{\kappa,\mu}$ is Whittaker's function and

$$\kappa = \frac{k_3 - 2ik_\perp^2}{2k_3} \tag{23}$$

$q = 1$

Equation (16) is in this case

$$t^2\ddot{F} + t\dot{F} + (-2ik_3 t^2 + k_\perp^2 + k_3^2 t^4)F = 0 \tag{24}$$

and the solutions are

$$F = \frac{1}{t}[A_k W_{1/2, ik_\perp/2}(ik_3 t^2) + B_k W_{-1/2, ik_\perp/2}(-ik_3 t^2)] \tag{25}$$

where $W_{\kappa,\mu}$ is Whittaker's function.

4. DISCUSSION

In this work we have been able to solve the Weyl equations in some members of a family of anisotropic spacetimes of Bianchi type I that are of some interest in cosmology. The exact solutions were obtained by means of a separation of variables that was possible because of the high degree of symmetry in this family of cosmological solutions. The second quantization, particle interpretation, and interaction with other fields are under consideration and will be reported in a future paper.

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REFERENCES

Audretsch, J., and Schäfer, G. (1978*a*). *General Relativity and Gravitation*, **9**, 243.
Audretsch, J., and Schäfer, G. (1978*b*). *General Relativity and Gravitation*, **9**, 489.
Barut, A. O., and Duru, I. H. (1987). *Physical Review D*, **36**, 3705.
Castagnino, M. A., El Hasi, C. D., Mazzitelli, F. D., and Paz, J. P. (1988). *Physics Letters A*, **128**, 25.
Chimento, L. P., and Mollerach, M. S. (1986). *Physical Review D*, **34**, 3689.
Chimento, L. P., and Mollerach, M. S. (1987). *Physics Letters A*, **121**, 7.
Iyer, B. R., and Kamaran, N. (1991). *Journal of Mathematical Physics*, **32**, 2497.
Iyer, B. R., and Vishveshwara, C. V. (1987). *Journal of Mathematical Physics*, **28**, 1377.
Jacobs, K. C. (1968). *Astrophysical Journal*, **153**, 661.
Krori, K. D., Borgohain, P., Karr, P. K., and Das, D. (1968). *Journal of Mathematical Physics*, **29**, 1645.
Leen, T. K., Parker, L., and Pimentel, L. O. (1983). *General Relativity and Gravitation*, **15**, 761.
Parker, L. (1980*a*). *Physical Review D*, **22**, 1922.
Parker, L. (1980*b*). *Physical Review Letters*, **44**, 1559.
Parker, L., and Pimentel, L. O. (1982). *Physical Review D*, **25**, 3180.
Percoco, U., and Villalba, V. M. (1991). *Canadian Journal of Physics*, **69**, 665.
Shishkin, G. V., and Andrushkevich, I. E. (1985). *Physics Letters*, **110A**, 84.
Shishkin, G. V., and Villalba, V. M. (1989). *Journal of Mathematical Physics*, **30**, 2132.
Thorne, K. S. (1967). *Astrophysical Journal*, **148**, 51.
Vajk, J. P., and Eltgroth, P. G. (1970). *Journal of Mathematical Physics*, **11**, 2212.
Villalba, V. M. (1990). *Journal of Mathematical Physics*, **31**, 1483.
Villalba, V. M., and Percoco, U. (1990). *Journal of Mathematical Physics*, **31**, 715.